Formal Verification of Safety Buffers for State-Based Conflict Detection

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Abstract

An approach for modeling uncertainty in aircraft position and surveillance information is proposed. The approach is used to provide an upper bound to the probability of missed alerts in state-based conflict detection algorithms. This bound yields an analytical definition of safety buffers that guarantees that, under well-defined assumptions on aircraft state information uncertainty, state-based conflict detection algorithms do not miss any conflicts. The results are presented as theorems, which were formally proven using a mechanical theorem prover.

1 Introduction

Advances in global positioning systems and communication technology enable air traffic concepts where the aircraft separation requirement relies on computer-based conflict detection and resolution (CD&R) systems [14]. In some of these concepts, the conflict management functionality is structured in several layers [24]. In the upper layers, strategic CD&R systems provide advance separation assurance functionality that takes into account long lookahead times, flight plans, special airspace restrictions, winds, and weather [13]. The lower layers typically deal with tactical decisions for short lookahead times. Since the lower layers provide the last line of defense in a multi-layered concept, tactical CD&R systems are considerably simpler and more efficient than strategic systems.

During recent years, several state-based tactical CD&R algorithms have been proposed [1, 6, 8, 11, 16]. State-based algorithms probe and solve conflicts by only using aircraft state information, i.e., the current position and velocity vectors of the aircraft, and a nominal mass-point model of aircraft trajectories. These assumptions allow for efficient implementations

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that rely on analytical methods. To accomodate for the difference between the actual aircraft trajectories and the predicted straight line trajectories used by these methods, it is generally assumed that state-based CD&R algorithms are frequently executed. Typically, state-based CD&R approaches are used in airborne concepts [11] where they are executed in each aircraft as frequently as position and surveillance information is updated, e.g., 1 Hz.

Given the safety-critical nature of tactical separation assurance systems, some state-based CD&R algorithms [6, 9, 15, 16] have been formally analyzed for safety properties such as *independence*, i.e., minimum separation is guaranteed when one of the aircraft maneuvers, and *implicit coordination*, i.e., minimum separation is guaranteed when both aircraft maneuver with no explicit coordination between them [7]. These safety properties highly depend on the assumption that aircraft state information is accurately known.

The position provided by global navigation satellite systems like Global Positioning System (GPS) is accurate up to a few meters (about 10m) [1] and surveillance information systems such as Automatic Dependent Surveillance-Broadcast (ADS-B) loses messages due to signal attenuation [23]. Errors in position and velocity negatively affect the safety performance of state-based CD&R systems. To mitigate these effects, state-based CD&R algorithms are used with safety buffers that increase the minimum separation distance between the aircraft. These safety buffers decrease the number of missed alerts but increase the number of false alert. Usually, appropriate values for safety buffers are determined by experimentation and simulation.

This paper presents the formal verification of safety buffers for conflict detection. Specific formulas are given for safety buffers, and a theorem is stated that represents a proved result that these formulas are correct and therefore satisfy a key probabilistic property. Section 2 contains formal defintions related to conflict detection algorithms. Section 3 models GPS and ADS-B errors with random variables on an arbitrary probability space. It is proved in that section that given random variables for positions and velocities, conflict between aircraft is also a random variable. Section 4 gives specific formulas for safety buffers for distance and time that can be used to provide upper bounds on the probability that a conflict detection algorithm will incorrectly miss a conflict. These formulas are then used to give safety buffers that guarantee that there are no missed conflicts, in the case where absolute bounds are known on position and velocity vectors for two aircraft. Finally, Section 5 presents a table that contains specific upper bounds on the probability that a correct conflict detection algorithm will miss a given conflict. This table depends on the document DO-242A [23], which specifies several system performance confidence-levels that are to be included in ADS-B messages detailing how precise and trusted the contained state information is.

The mathematical development presented in this paper has been specified and formally verified in the Prototype Verification System (PVS) [17]. PVS is a proof assistant that consists of a specification language, based on classical higher-order logic, and a mechanical theorem prover for this logic. The PVS specification language allows for the precise definition of mathematical objects such as *functions* and *relations*, and the precise statement of logical formulas such as *lemmas* and *theorems*. Proofs of logical formulas can be mechanically

[1] See `http://www.kowoma.de/en/gps/errors.htm`.

checked using the PVS theorem prover, which guarantees that every proof step is correct and that all possible cases of a proof are covered. All lemmas and theorems presented in this paper have been mechanically checked in PVS. For the sake of simplicity, only proof sketches of the main results are presented in the paper. The development presented here, including all definitions and formal proofs, is part of the Airborne Coordinated Conflict Detection and Resolution (ACCoRD) framework [16].

The use of a formal language, e.g., in this case the specification language of PVS, enforces rigorous definitions of mathematical objects, where all dependencies are clearly specified. This level of rigor guarantees a very high confidence on the correctness of the results presented in this paper. However, this also makes the notation heavy and difficult to read for the non-expert reader. For this reason, the work presented here uses standard mathematical notation and does not assume that the reader is familiar with the syntax or semantics of the PVS language.

2 State-Based Conflict Detection

Pairwise state-based conflict detection systems use the state information of two aircraft, which here are referred to as the *ownship* and the *intruder*, to detect conflicts between them. The state information for an aircraft includes its current position and velocity, and these are represented by points and vectors in a Cartesian coordinate system. For simplicity, this paper considers the two-dimensional space $\mathbb{R}^2$. However, the results can be extended to $\mathbb{R}^3$.

Aircraft trajectories are represented by a point moving at constant linear speed. The vectors $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i$, and $\mathbf{v}_i$ will be used to represent the ownship's current position and velocity and the intruder's current position and velocity (at time $t = 0$), respectively. Thus, the states of the ownship and the intruder at time t are given by $\mathbf{s}_o + t\,\mathbf{v}_o$ and $\mathbf{s}_i + t\,\mathbf{v}_i$, respectively. In later sections, $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i$, and $\mathbf{v}_i$ denote random variables with values in $\mathbb{R}^2$ to account for uncertainty in these vectors.

Under nominal operations, aircraft are required to maintain a certain separation. In the two-dimensional airspace, the separation requirement is specified by a minimum horizontal distance D. A *conflict* between the ownship and the intruder aircraft occurs when there is a time $t \in [0, T]$ at which the horizontal distance between the aircraft is projected to be less than D, i.e.,

$$\|(\mathbf{s}_o + t\,\mathbf{v}_o) - (\mathbf{s}_i + t\,\mathbf{v}_i)\| < D.$$

The time T is called the *lookahead* time. Typical values for D and T are 5 nautical miles and 5 minutes, respectively. In this paper these values are considered to be parameters.

Since $(\mathbf{s}_o + t\,\mathbf{v}_o) - (\mathbf{s}_i + t\,\mathbf{v}_i) = (\mathbf{s}_o - \mathbf{s}_i) + t\,(\mathbf{v}_o - \mathbf{v}_i)$, the predicate that characterizes conflicts can be defined in terms of the relative vectors $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, i.e., the relative position and velocity vectors, respectively, of the ownship with respect to the intruder. The predicate *conflict?*, which has as parameters the horizontal distance D, the lookahead time T, and the relative position and velocity of the aircraft, is formally defined as follows.

$$\mathit{conflict?}(D, T, \mathbf{s}, \mathbf{v}) \equiv \exists\, t \in [0, T] : \|\mathbf{s} + t\,\mathbf{v}\| < D.$$

A conflict detection system is an algorithm that computes whether the predicate *conflict?* holds for the actual states of two aircraft. A pairwise approach to conflict detection is assumed where each aircraft uses a conflict detection algorithm. The approach proposed in this paper takes the point of view of the ownship. However, the situation is symmetric from the point of view of the intruder aircraft. Formally, a two-dimensional conflict detection algorithm is defined as a function `cd` with parameters D and T, written in subscript, that takes as arguments the state information of two aircraft. It returns a value in $\mathbb{B} = \{\texttt{True}, \texttt{False}\}$ that represents whether or not a conflict has been detected.

The state information used by a conflict detection algorithm is provided by positioning and surveillance systems such as GPS and ADS-B. In order to distinguish the actual states of the aircraft, represented by $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i, \mathbf{v}_i$, from the measured states provided by these systems, the measured position and velocity of the ownship and intruder aircraft will be represented by the vectors $\mathbf{s_o^m}, \mathbf{v_o^m}$ and $\mathbf{s_i^m}, \mathbf{v_i^m}$, respectively.

Since conflict detection algorithms are safety critical applications, it is imperative that they compute an answer that is trustworthy. A conflict detection algorithm is said to be *correct* if in the absence of measurement errors the algorithm does not issue false alerts and does not miss any alerts. Formally, a conflict detection algorithm `cd` is correct if for all vectors $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i, \mathbf{v}_i, \mathbf{s_o^m}, \mathbf{v_o^m}, \mathbf{s_i^m}$, and $\mathbf{v_i^m}$, with $\mathbf{s_o^m} = \mathbf{s}_o, \mathbf{v_o^m} = \mathbf{v}_o, \mathbf{s_i^m} = \mathbf{s}_i$, and $\mathbf{v_i^m} = \mathbf{v}_i$, then

$$\texttt{cd}_{D,T}(\mathbf{s_o^m}, \mathbf{v_o^m}, \mathbf{s_i^m}, \mathbf{v_i^m}) = \texttt{True} \iff \textit{conflict?}(D, T, \mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i). \tag{2.1}$$

In theory, conflict detection algorithms are designed to be correct, e.g., the conflict detection algorithm `CD2D`, which is part of NASA's Airborne Coordinated Conflict Resolution and Detection (ACCoRD) framework [16], satisfies this property. In practice, the existence of uncertainty in surveillance information implies that the equalities $\mathbf{s_o^m} = \mathbf{s}_o, \mathbf{v_o^m} = \mathbf{v}_o, \mathbf{s_i^m} = \mathbf{s}_i$, and $\mathbf{v_i^m} = \mathbf{v}_i$ may not hold. Thus, conflict detection algorithms, including correct algorithms such as `CD2D`, detect conflicts with inexact information, and they can therefore have false and missed alerts. Therefore, CD&R algorithms are generally used with slightly increased D and T values to accommodate for state information uncertainty. The added values are called *safety buffers*, and their sizes are often determined by experimentation and simulation.

Increasing the size of these safety buffers will reduce number of missed alerts. However, as the size of the safety buffers increases, the number of false alerts increases as well. Missed alerts are an obvious cause of safety concerns. False alerts have also safety implications as they may diminish the confidence that crew members and air traffic controllers have on the separation assurance logic. Appropriate choices of safety buffers are crucial to the safety performance of a conflict detection system.

This paper provides analytical definitions of safety buffers and sufficient conditions under which correct conflict detection algorithms can be used without missing alerts. More precisely, definitions of positive numbers ψ and λ are provided such that under well-defined hypotheses on the information uncertainty, it can be proved that

$$\textit{conflict?}(D, T, \mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i) \Longrightarrow \texttt{cd}_{D+\psi,T+\lambda}(\mathbf{s_o^m}, \mathbf{v_o^m}, \mathbf{s_i^m}, \mathbf{v_i^m}) = \texttt{True}.$$

3 State Information Uncertainty

This paper considers two kinds of uncertainties: uncertainty due to measurement errors in global positioning systems such as GPS, and uncertainty due to infrequent traffic information updates from surveillance systems such as ADS-B. Concretely, state information uncertainty is modeled through random variables that represent measurement errors due to (1) GPS position inaccuracy and (2) dropped ADS-B messages. Here, GPS and ADS-B are used for illustration purposes. The approach presented here could be adapted for uncertainty due to devices other than GPS and broadcast methods other than ADS-B.

Recall that a random variable is a function $f\colon \Omega \to X$, where $(\Omega, \sigma(\Omega))$ is a probability space, i.e., Ω is a set, $\sigma(\Omega)$ is a σ-algebra on the set Ω (a set of subsets of Ω), and there is a probability function P that maps elements of $\sigma(\Omega)$ to probabilities in the interval $[0, 1]$; cf. [21]. The set X is any measure space, and the function f must be *measurable*, in the sense of real analysis [22]. In what follows, the same probability space $(\Omega, \sigma(\Omega))$ will be used to model all of the random variables, e.g., GPS inaccuracy, dropped ADS-B message, conflict detection, etc. This is mathematically valid because even if two random variables are modeled with different probability spaces for their respective domains, equivalent random variables can be constructed whose domains are the *same* probability space. In fact, any random variable has an equivalent representation as a random variable with domain given by the uniform distribution on the interval $[0, 1]$; cf. [5].

Given a subset S of Ω such that $S \in \sigma(\Omega)$, the probability function P gives the *probability* $P(S)$ of S. Any random variable $f\colon \Omega \to X$ induces a probability *Prob* on measurable subset of X that is defined by $\mathit{Prob}(Y) = P(\{\chi \in \Omega \mid f(\chi) \in Y\})$, where Y is measurable. In addition, if X is a subset of the real numbers, then it is standard notation to define $\mathit{Prob}(f \geq r) = P(\{\chi \in \Omega \mid f(\chi) \geq r\})$ for $r \in \mathbb{R}$.

3.1 Modeling Uncertainty with Random Variables

Each aircraft uses GPS to determine its current state, i.e., its position, $\mathbf{s}_o$ or $\mathbf{s}_i$, and velocity vector, $\mathbf{v}_o$ or $\mathbf{v}_i$. ADS-B broadcasts this information to the airspace at regular intervals, and the interval between ADS-B broadcasts will be denoted α. Typically, the ADS-B system will be configured to broadcast this information once per second, i.e., $\alpha = 1$ second. Due to signal attenuation, it is possible that several consecutive position and velocity updates from the intruder have been dropped and were therefore not received by the ownship. This results in greater uncertainty in the values of the intruder's current state, i.e., $\mathbf{s}_i$ and $\mathbf{v}_i$. ADS-B message loss due to signal attenuation can be modeled as random variable:

$$\mathcal{A}\colon \Omega \to \mathbb{N},$$

where $(\Omega, \sigma(\Omega))$ is a probability space. The random variable $\mathcal{A}$ returns the number of consecutive ADS-B messages from the intruder that were not received by the ownship, since the last received message from the intruder. It is important to note that the length of time since the last ADS-B update from the intruder was received by the ownship is given by multiplying the return value of $\mathcal{A}$ by α. This length of time is modeled by the random variable $\Upsilon\colon \Omega \to \mathbb{R}$ that maps $\chi \in \Omega$ into $\alpha\mathcal{A}(\chi)$, where the units of the domain are implicitly the units of α.

Standard inaccuracies in GPS position predictions, which are also used to predict velocities, imply that the measured positions $\mathbf{s_o^m}, \mathbf{s_i^m}$ and velocities $\mathbf{v_o^m}, \mathbf{v_i^m}$ may have errors. Thus, the actual positions $\mathbf{s}_o, \mathbf{s}_i$ and velocities $\mathbf{v}_o, \mathbf{v}_i$ are all modeled as random variables from Ω to $\mathbb{R}^2$:

$$\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i\colon \Omega \to \mathbb{R}^2.$$

The vectors $\mathbf{s_i^m}$ and $\mathbf{v_i^m}$ represent the intruder's reported position and velocity vectors, respectively, from the *last* ADS-B signal that was received by the ownship, and the vectors $\mathbf{s_o^m}$ and $\mathbf{v_o^m}$ represent the ownship's measured position and velocity at that time. If the current time is $t = 0$, then the time at which $\mathbf{s_o^m}, \mathbf{s_i^m}, \mathbf{v_o^m}, \mathbf{v_i^m}$ were measured is given by the random variable Υ. Thus, if it is known that there are no errors in the measurements $\mathbf{s_o^m}, \mathbf{s_i^m}, \mathbf{v_o^m}, \mathbf{v_i^m}$, then the equalities $\mathbf{s}_o - \Upsilon\mathbf{v}_o = \mathbf{s_o^m}$, $\mathbf{s}_i - \Upsilon\mathbf{v}_i = \mathbf{s_i^m}$, $\mathbf{v}_o = \mathbf{v_o^m}$, and $\mathbf{v}_i = \mathbf{v_i^m}$ all hold *as random variables* $\Omega \to \mathbb{R}^2$. The random variables Υ, $\mathbf{v}_o$, and $\mathbf{v}_i$ have units given by time, speed, and speed, respectively.

This paper focuses on the case where there are possible errors in the measurements $\mathbf{s_o^m}, \mathbf{s_i^m}, \mathbf{v_o^m}, \mathbf{v_i^m}$, modeled by the random variables $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i$. In this case, the absolute values (i.e. errors) $|(\mathbf{s}_o - \Upsilon\mathbf{v}_o) - \mathbf{s_o^m}|$, $|(\mathbf{s}_i - \Upsilon\mathbf{v}_i) - \mathbf{s_i^m}|$, $|\mathbf{v}_o - \mathbf{v_o^m}|$, and $|\mathbf{v}_i - \mathbf{v_i^m}|$ are all random variables $\Omega \to \mathbb{R}_{\geq 0}$, and they therefore induce probabilites on subsets of $\mathbb{R}_{\geq 0}$, respectively. Thus, in the following sections, the probabilites

$$\begin{aligned} \mathit{Prob}(\|(\mathbf{s}_o - \Upsilon\mathbf{v}_o) - \mathbf{s_o^m}\|| \geq a_o), \\ \mathit{Prob}(\|(\mathbf{s}_i - \Upsilon\mathbf{v}_i) - \mathbf{s_i^m}\| \geq a_i), \\ \mathit{Prob}(\|\mathbf{v}_o - \mathbf{v_o^m}\| \geq b_o), \\ \mathit{Prob}(\|\mathbf{v}_i - \mathbf{v_i^m}\| \geq b_i) \end{aligned}$$

will be used to bound the effects of GPS measurement errors on conflict detection. Here, the distances a_o and a_i and the speeds b_o and b_i are standardized navigation accuracy parameters. For instance, $a_o = a_i = 30$ m and $b_o = b_i = 0.3$ m/s correspond to navigation accuracy categories NAC_P 9 and NAC_V 4, respectively, as specified in [23]. This specification is for 95 percent confidence intervals on the position and velocity vectors of aircraft, within the given ranges. Other choices for a_o, a_i, b_o, and b_i may be considered, and thus in the next sections they are simply treated as parameters.

Finally, given D and T, a conflict between the ownship and the intruder will be modeled as the random variable $\mathcal{C}_{D,T} : \Omega \to \mathbb{B}$ that maps $\chi \in \Omega$ into *conflict?*$(D, T, \mathbf{s}_o(\chi) - \mathbf{s}_i(\chi), \mathbf{v}_o(\chi) - \mathbf{v}_i(\chi))$.

The fact that the function $\mathcal{C}_{D,T}$ is a random variable is not immediately obvious. In fact, if κ is any boolean function on four vectors, it is not necessarily true that $\kappa(\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i, \mathbf{v}_i)$ is a random variable on Ω. While it is true that scalar multiples, sums, dot products, cross products, etc. of random variables $\Omega \to \mathbb{R}^2$ are also random variables, the definition of the random variable $\mathcal{C}_{D,T}$ involves an existential quantifier (i.e. $\exists$) in the definition of *conflict?*. However, the following lemma has been formally proved in PVS.

Lemma 1 *The boolean function $\mathcal{C}_{D,T}$ is a random variable on Ω.*

Proof: The conflict detection algorithm `CD2D` is equivalent to the predicate *conflict?* (the proof of this fact is provided in the PVS formal development available from [16]). Thus, the predicate *conflict?* can be replaced by `CD2D` in the definition of $\mathcal{C}_{D,T}$ without changing the function. It therefore suffices to show that the function that maps an element χ of Ω to $\texttt{CD2D}_{D,T}(\mathbf{s}_o(\chi), \mathbf{v}_o(\chi), \mathbf{s}_i(\chi), \mathbf{v}_i(\chi))$ is a random varible. This expression is of the form

$$\texttt{CD2D}_{D,T}(\mathbf{s}_o(\chi), \mathbf{v}_o(\chi), \mathbf{s}_i(\chi), \mathbf{v}_i(\chi)) = \begin{cases} f(\chi) & \text{if } g(\chi) = 0, \\ h(\chi) & \text{if } g(\chi) / = 0, \end{cases}$$

where $f, h \colon \Omega \to \mathbb{B}$ and $g \colon \Omega \to \mathbb{R}$ are all random variables. This function is equal to $f \cdot \texttt{Char}_E + h \cdot \texttt{Char}_{\neg E}$, where $E = \{\chi \in \Omega \,|\, g(\chi) = 0\}$, $\neg E$ is the complement of E, and `Char` denotes the characteristic function of a given set. Since the function g is a random variable, E and $\neg E$ are by definition elements of $\sigma(\Omega)$, and so their characteristic functions are random variables. Hence, the function $f \cdot \texttt{Char}_E + h \cdot \texttt{Char}_{\neg E}$ is a sum of products of random variables, and it is therefore a random variable as well. $\square$

Since $\mathcal{C}_{D,T}$ is a random variable, the probability that the two aircraft are actually in conflict is formally defined as

$$\mathit{Prob}(\mathit{conflict?}(D, T, \mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)) = P(\{\chi \in \Omega \,|\, \mathcal{C}_{D,T}(\chi) = \texttt{True}\}).$$

3.2 Distribution of the ADS-B Random Variable

Under the assumption that there is no ADS-B signal interference due to multiple intruder aircraft, the distribution of ADS-B message loss $\mathcal{A}$ follows a Poisson distribution as discussed in [3]. The probability that a given ADS-B message from the intruder aircraft will not be received by the ownship, which is equal to $p(\{0\})$, is (approximately) given by $1 - \left(\frac{r}{r_0}\right)^k$ with $r \leq r_0$, where $k = 6.4314$ and $r_0 = 96.6$ nmi [3]. The number r is the current distance between the two aircraft. Thus, if it is known that the ownship and the intruder are no greater than 60 nmi apart, a reasonable distance for most commercial aircraft given short lookahead times such as 3 minutes, then the probability that a given message will be received is bounded below by 0.953, where in the formal language of random variables, this is expressed as $P(\{\chi \in \Omega \,|\, \mathcal{A}(\chi) = 0\}) \geq 0.953$. The specific probability 0.953 is not critical to the constructions in this paper. Thus, the probability that a given ADS-B message sent by the intruder will be received by ownship will be denoted by the variable η:

$$\eta = P(\{\chi \in \Omega \,|\, \mathcal{A}(\chi) = 0\}).$$

The key assumption that can be used to deduce that $\mathcal{A}$ follows a Poisson distribution is that whether any particular ADS-B message from the intruder aircraft is received by the ownship is independent from whether any other, different, ADS-B message from the intruder is received, for $i \geq 0$. This implies that for each $i \geq 0$, the probability that the last ADS-B message sent by the intruder that was received by the ownship was the $i+1$-st message ago (sent $\alpha \cdot i$ in the past) is given by

$$P(A_i) = \eta(1-\eta)^i, \tag{3.2}$$

where

$$A_i = \{\chi \in \Omega \,|\, \mathcal{A}(\chi) = i\}.$$

This is because the last i messages (sent $0, \alpha 1, \ldots$ and $\alpha i - 1$ seconds ago) have been dropped, which has a probability of $(1-\eta)^i$ of occurring, and the message sent exactly i-seconds ago was not dropped, which has a probability of η of occurring.

4 Safety Buffers

As noted in previous sections, the correctness of a conflict detection algorithm $\mathtt{cd}_{D,T}$ can be affected by errors in GPS measurements or delays in ADS-B message updates. To counteract the effects of these errors on the conflict detection probe `cd`, a positive distance ψ and a positive time λ can be artificially added to the distance D and the time T when they are used as parameters in `cd`. That is, to make the algorithm more likely to return `True`, the parameters $D+\psi$ and $T+\lambda$ can be used in place of D and T in the algorithm `cd`. The distance ψ and the time λ are called safety buffers because the algorithm $\mathtt{cd}_{D+\psi,T+\lambda}$ is more likely to return `True` than $\mathtt{cd}_{D,T}$, and hence they are more conservative from a safety standpoint.

4.1 Probability of Conflict

Given the use of safety buffers ψ and λ in the conflict detection algorithm `cd`, as described above, a missed alert occurs when $\mathtt{cd}_{D+\psi,T+\lambda}(\mathbf{s^m_o}, \mathbf{v^m_o}, \mathbf{s^m_i}, \mathbf{v^m_i})$ returns `False` but the aircraft are actually in conflict. So an upper bound for the probability of a missed alert is actually an upper bound on the probability *Prob*(*conflict?*$(D, T, \mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)$) that the aircraft are actually in conflict (cf. Section 3.1). Define $\mathcal{G}$ to be the set of $\chi \in \Omega$ where at least one of the following inequalities holds.

$$\begin{aligned}
\|(\mathbf{s}_o(\chi) - \alpha\mathcal{A}(\chi)\mathbf{v}_o(\chi)) - \mathbf{s^m_o}\|| &\geq a_o, \\
\|(\mathbf{s}_i(\chi) - \alpha\mathcal{A}(\chi)\mathbf{v}_i(\chi)) - \mathbf{s^m_i}\| &\geq a_i, \\
\|\mathbf{v}_o(\chi) - \mathbf{v^m_o}\| &\geq b_o, \\
\|\mathbf{v}_i(\chi) - \mathbf{v^m_i}\| &\geq b_i
\end{aligned}$$

Define $\mathcal{T} = \{\chi \in \Omega \,|\, \mathcal{C}_{D,T}(\chi) = \mathtt{True}\}$. Note that the set Ω decomposes as an infinite union of pairwise disjoint sets $\Omega = \bigcup_{i=0}^{\infty} A_i$, where A_i is defined in Section 3.2. Recall that for a given set Z, Z^c denotes the complement of Z. Then standard probabilistic manipulations can be used to show that the probability *Prob*(*conflict?*$(D, T, \mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)$) decomposes as an infinite sum as follows.

$$\begin{aligned}
\mathit{Prob}(\mathit{conflict?}(D,T,\mathbf{s}_o-\mathbf{s}_i,\mathbf{v}_o-\mathbf{v}_i)) &= P(\mathcal{T}) \\
&= P(\mathcal{T}\cap\mathcal{G}) + P(\mathcal{T}\cap\mathcal{G}^c) \\
&= P(\mathcal{T}\cap\mathcal{G}) + P\left(\bigcup_{i=0}^{\infty}(\mathcal{T}\cap A_i\cap\mathcal{G}^c)\right) \qquad (4.3)\\
&= P(\mathcal{T}\cap\mathcal{G}) + \sum_{i=0}^{\infty} P(\mathcal{T}\cap A_i\cap\mathcal{G}^c).
\end{aligned}$$

This equation implies that if d is any integer (a specific number of messages), then

$$\begin{aligned}
\mathit{Prob}(\mathit{conflict?}(D,T,\mathbf{s}_o-\mathbf{s}_i,\mathbf{v}_o-\mathbf{v}_i)) &= P(\mathcal{T}\cap\mathcal{G}) \ + \ \sum_{i=0}^{\infty} P(\mathcal{T}\cap A_i\cap\mathcal{G}^c) \\
&\le P(\mathcal{G}) \ + \ \sum_{i=0}^{\infty} P(\mathcal{T}\cap A_i\cap\mathcal{G}^c) \\
&\le P(\mathcal{G}) \ + \ \sum_{i=0}^{d} P(A_i) \ + \ P(\mathcal{T}\cap A_i\cap\mathcal{G}^c) \qquad (4.4)\\
&= P(\mathcal{G}) \ + \ \sum_{i=d+1}^{\infty} \eta(1-\eta)^i \ + \ \sum_{i=0}^{d} P(\mathcal{T}\cap A_i\cap\mathcal{G}^c) \\
&= P(\mathcal{G}) \ + \ (1-\eta)^{d+1} \ + \ \sum_{i=0}^{d} P(\mathcal{T}\cap A_i\cap\mathcal{G}^c)
\end{aligned}$$

The number d can be chosen so that the finite sum is a good approximation to the infinite sum (since $(1-\eta)^{d+1}$ is quite small). This equation is true for any choice of d. Formula (4.4) has been formally proved in PVS and can be found in the ACCoRD development at [16].

4.2 Probability of a Missed Alert

Suppose now that confidence intervals are known for the accuracy of the random variables $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o$, and $\mathbf{v}_i$. That is, suppose that probabilities p_{so}, p_{vo}, p_{si}, and p_{vi} are known such that

$$\begin{aligned}
\mathit{Prob}(\|(\mathbf{s}_o-\Upsilon\mathbf{v}_o)-\mathbf{s}_\mathbf{o}^\mathbf{m}\|| \ge a_o) &\le p_{so}, \\
\mathit{Prob}(\|(\mathbf{s}_i-\Upsilon\mathbf{v}_i)-\mathbf{s}_\mathbf{i}^\mathbf{m}\| \ge a_i) &\le p_{si}, \\
\mathit{Prob}(\|\mathbf{v}_o-\mathbf{v}_\mathbf{o}^\mathbf{m}\| \ge b_o) &\le p_{vo}, \\
\mathit{Prob}(\|\mathbf{v}_i-\mathbf{v}_\mathbf{i}^\mathbf{m}\| \ge b_i) &\le p_{vi}.
\end{aligned}$$

It follows immediately that

$$P(\mathcal{G}) \le p_{so} + p_{si} + p_{vo} + p_{vi}. \qquad (4.5)$$

Examples of such bounds p_{so}, p_{vo}, p_{si}, and p_{vi} on these probabilities can be found in DO-242A [23], which specifies several system performance confidence-levels that are to be included in ADS-B messages, and details how precise and trusted the contained state information is.

Formulas (4.4) and (4.5) imply that if $P(\mathcal{T} \cap A_i \cap \mathcal{G}^c) = 0$ for $i \leq d$, then the probability that $\texttt{cd}_{D,T}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i) = \texttt{True}$ is bounded above by $p_{so} + p_{si} + p_{vo} + p_{vi} + (1-\eta)^{d+1}$. The following lemma presents particular choices of the safety buffers ψ and λ that can be used to ensure that this bound is satisfied. The lemma refers to the distances a_o and a_i and the speeds b_o and b_i that define the probabilities $p_{so}, p_{vo}, p_{si}, p_{vi}$. It also uses the time α, which is the regular interval at which ADS-B messages are sent by the intruder aircraft. The following lemma has been formally proved in PVS.

Lemma 2 *Let $\mathbf{s}^m = \mathbf{s^m_o} - \mathbf{s^m_i}$, $\mathbf{v}^m = \mathbf{v^m_o} - \mathbf{v^m_i}$. For any integer $d \geq 0$, if*

1. $\|\mathbf{v}^m\| > b_o + b_i$,
2. $\ell = (\|\mathbf{s}^m\| + a_o + a_i + \alpha d \cdot (\|\mathbf{v}^m\| + b_o + b_i))/(\|\mathbf{v}^m\| - b_o - b_i)$,
3. $\psi = a_o + a_i + (\min(T, \ell) + \alpha d)(b_o + b_i)$, *and*
4. $\texttt{cd}_{D+\psi,T+\alpha d}(\mathbf{s^m_o}, \mathbf{v^m_o}, \mathbf{s^m_i}, \mathbf{v^m_i}) = \texttt{False}$,

then, for $j \in \{0, \ldots, d\}$,

$$P(\mathcal{T} \cap A_j \cap \mathcal{G}^c) = 0.$$

Proof: It suffices to prove that, given the hypotheses of this lemma, $\mathcal{T} \cap A_j \cap \mathcal{G}^c$ is empty. Suppose by way of contradiction that $\chi \in \mathcal{T} \cap A_j \cap \mathcal{G}^c$. Since $\chi \in \mathcal{T}$, it follows that $\texttt{cd}_{D,T}(\mathbf{s}_o(\chi), \mathbf{s}_i(\chi), \mathbf{v}_o(\chi), \mathbf{v}_i(\chi)) = \texttt{True}$. Since $\chi \in A_j$, $\mathcal{A}(\chi) = j$. Finally, since $\chi \in \mathcal{G}^c$, the equations $\|(\mathbf{s}_o(\chi) - \alpha j \mathbf{v}_o(\chi)) - \mathbf{s^m_o}\| < a_o$, $\|(\mathbf{s}_i(\chi) - \alpha j \mathbf{v}_i(\chi)) - \mathbf{s^m_i}\| < a_i$, $\|\mathbf{v}_o(\chi) - \mathbf{v^m_o}\| < b_o$, and $\|\mathbf{v}_i(\chi) - \mathbf{v^m_i}\| < b_i$ are all satisfied.

As in Section 2, let $\mathbf{s}$ and $\mathbf{v}$ denote the relative position and velocities $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$. It is easy to see that $\|\mathbf{s}(\chi) + t\mathbf{v}(\chi)\|^2$ is a quadratic in t that attains its minimum at $t = -\mathbf{s}(\chi) \cdot \mathbf{v}(\chi)/\|\mathbf{v}(\chi)\|^2$. Thus, the fact that $\texttt{cd}_{D,T}(\mathbf{s}_o(\chi), \mathbf{s}_i(\chi), \mathbf{v}_o(\chi), \mathbf{v}_i(\chi)) = \texttt{True}$ (since $\chi \in \mathcal{T}$) implies that there exists $t^* \in [0, \min(T, -\mathbf{s}(\chi) \cdot \mathbf{v}(\chi)/\|\mathbf{v}(\chi)\|^2)]$ such that $\|\mathbf{s}(\chi) + t^*\mathbf{v}(\chi)\| < D$. Then $t^* + \alpha j \in [0, \min(T, -\mathbf{s}(\chi) \cdot \mathbf{v}(\chi)/\|\mathbf{v}(\chi)\|^2) + \alpha d]$ and since $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, it suffices to show that $\|\mathbf{s}^m + (t^* + \alpha j)\mathbf{v}^m\| < \psi + D$, which is a contradiction, since $\texttt{cd}_{D+\psi,T+\alpha d}(\mathbf{s^m_o}, \mathbf{v^m_o}, \mathbf{s^m_i}, \mathbf{v^m_i}) = \texttt{False}$. If it can be proved that

$t^* + \alpha j \le \min(T, \ell) + \alpha d$, then the result will follow, since

$$\begin{aligned}
&\|\mathbf{s}^m + (t^* + \alpha j)\mathbf{v}^m\| \\
&= \|(\mathbf{s_o^m} - \mathbf{s_i^m}) + (t^* + \alpha j)(\mathbf{v_o^m} - \mathbf{v_i^m})\| \\
&= \|(\mathbf{s_o^m} - \mathbf{s_i^m}) + (t^* + \alpha j)(\mathbf{v_o^m} - \mathbf{v_i^m}) - (\mathbf{s}(\chi) + t^*\mathbf{v}(\chi)) + (\mathbf{s}(\chi) + t^*\mathbf{v}(\chi))\| \\
&= \|(\mathbf{s_o^m} - (\mathbf{s}_o(\chi) - \alpha j \mathbf{v}_o(\chi))) - (\mathbf{s_i^m} - (\mathbf{s}_i(\chi) - \alpha j \mathbf{v}_i(\chi))) + (t^* + \alpha j)(\mathbf{v_o^m} - \mathbf{v}_o(\chi)) \\
&\qquad - (t^* + \alpha j)(\mathbf{v_i^m} - \mathbf{v}_i(\chi)) + (\mathbf{s} + t^*\mathbf{v}(\chi))\| \\
&\le \|\mathbf{s_o^m} - (\mathbf{s}_o(\chi) - \alpha j \mathbf{v}_o(\chi))\| + \|\mathbf{s_i^m} - (\mathbf{s}_i(\chi) - \alpha j \mathbf{v}_i(\chi))\| \\
&\qquad + (t^* + \alpha j)\|\mathbf{v_o^m} - \mathbf{v}_o(\chi)\| + (t^* + \alpha j)\|\mathbf{v_i^m} - \mathbf{v}_i(\chi)\| + \|\mathbf{s} + t^*\mathbf{v}(\chi))\| \\
&= \|\mathbf{s_o^m} - (\mathbf{s}_o(\chi) - \alpha \mathcal{A}(\chi) \mathbf{v}_o(\chi))\| + \|\mathbf{s_i^m} - (\mathbf{s}_i(\chi) - \alpha \mathcal{A}(\chi) \mathbf{v}_i(\chi))\| \\
&\qquad + (t^* + \alpha j)\|\mathbf{v_o^m} - \mathbf{v}_o(\chi)\| + (t^* + \alpha j)\|\mathbf{v_i^m} - \mathbf{v}_i(\chi)\| + \|\mathbf{s} + t^*\mathbf{v}(\chi))\| \\
&< a_o + a_i + (t^* + \alpha j)b_o + (t^* + \alpha j)b_i + D \\
&\le a_o + a_i + (t^* + \alpha d)(b_o + b_i) + D \\
&\le \psi + D.
\end{aligned}$$

Since $t^* + \alpha j \in [0, \min(T, -\mathbf{s}(\chi) \cdot \mathbf{v}(\chi)/\|\mathbf{v}(\chi)\|^2) + \alpha d]$ and $\alpha j \le \alpha d$, it therefore suffices to prove that $-\mathbf{s}(\chi) \cdot \mathbf{v}(\chi)/\|\mathbf{v}(\chi)\|^2 \le \ell$. The Cauchy-Schwartz inequality implies that $-\mathbf{s}(\chi)\cdot\mathbf{v}(\chi) \le \|\mathbf{s}(\chi)\|\cdot\|\mathbf{v}(\chi)\|$, so it suffices to prove that $\|\mathbf{s}(\chi)\|/\|\mathbf{v}(\chi)\| \le \ell$. This inequality can be verified by proving the following two inequalities.

$$\|\mathbf{s}(\chi)\| \le \|\mathbf{s}^m\| + a_o + a_i + \alpha d \cdot (\|\mathbf{v}^m\| + b_o + b_i)$$

$$\|\mathbf{v}(\chi)\| \ge \|\mathbf{v}^m\| - b_o - b_i$$

These two formulas follow from basic applications of the triangle inequality and the facts that $\|(\mathbf{s}_o(\chi) - \alpha j \mathbf{v}_o(\chi)) - \mathbf{s_o^m}\|\| < a_o$, $\|(\mathbf{s}_i(\chi) - \alpha j \mathbf{v}_i(\chi)) - \mathbf{s_i^m}\| < a_i$, $\|\mathbf{v}_o(\chi) - \mathbf{v_o^m}\| < b_o$, $\|\mathbf{v}_i(\chi) - \mathbf{v_i^m}\| < b_i$, and $\alpha j \le \alpha d$. $\square$

The following theorem uses Lemma 2 to give an upper bound on the probability of a missed alert, if the safety buffers ψ and αd given in that lemma are used. This theorem follows trivially from that lemma and Formula (4.4) and has also been proved in PVS.

Theorem 1 *Let* $\mathbf{s}^m = \mathbf{s_o^m} - \mathbf{s_i^m}$, $\mathbf{v}^m = \mathbf{v_o^m} - \mathbf{v_i^m}$. *Suppose that*

1. $d \ge 0$, $\|\mathbf{v}^m\| > b_o + b_i$
2. $\ell = (\|\mathbf{s}^m\| + a_o + a_i + \alpha d \cdot (\|\mathbf{v}^m\| + b_o + b_i))/(\|\mathbf{v}^m\| - b_o - b_i)$
3. $\psi = a_o + a_i + (\min(T, \ell) + \alpha d)(b_o + b_i)$
4. $\lambda = \alpha d$
5. $\mathtt{cd}_{D+\psi, T+\lambda}(\mathbf{s_o^m}, \mathbf{v_o^m}, \mathbf{s_i^m}, \mathbf{v_i^m}) = \mathtt{False}$.

The probability of a missed alert, i.e. that $\mathbf{cd}_{D,T}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i) = \texttt{True}$*, is no greater than* $p_{so} + p_{vo} + p_{si} + p_{vi} + (1-\eta)^{d+1}$*, where* η *is the the probability that a given ADS-B message sent by the intruder will be received by ownship.*

A *missed alert* is a conflict that is not detected. Artificially increasing the distance D and the lookahead time T in the conflict probe `cd` will make missed alerts less likely. The theorem above gives specific formulas for safety buffers that can be used to ensure that the probability of a missed alert is sufficiently small. The speeds b_o and b_i, and the probabilities p_{so}, p_{vo}, p_{si} and p_{vi} are variables in this theorem and can be changed based on the application. Formula (4.5) expresses the relationships between $a_o, a_i, b_o, b_i, p_{so}, p_{vo}, p_{si}$ and p_{vi}. Given these inputs, the associated upper bound for the probability of a missed alert is

$$p_{missed-alert} = p_{so} + p_{vo} + p_{si} + p_{vi} + (1-\eta)^{d+1}. \tag{4.6}$$

In the equation above, the amount ψ that D should be artificially increased to ensure that the probability of a missed alert is less than $p_{missed-alert}$ is given by

$$\psi = a_o + a_i + (\min(T, \ell) + \alpha d)(b_o + b_i), \tag{4.7}$$

where as above, α denotes the time period between consecutive ADS-B broadcasts by the intruder aircraft. It should be noted that Formulas (4.7) and (4.6) imply that if the velocity b dominates the calculation of ψ, then as ψ increases, d increases as well, and so the probability of a missed alert decreases.

The following is a formulation of Theorem 1 in PVS. The purpose of including the statement here is not technical, but rather so that the reader can conceptualize what is meant by a statement that is proved in PVS. The specifics of the PVS notation are unimportant, so most of the technical details are omitted.

```
Theorem1 : THEOREM
    sm = som-sim AND vm = vom-vim AND norm(vm)>bo+bi AND
    P(GsetPosition(so,som,vo,alpha,A,ao)) <= prso AND
    P(GsetVelocity(vo,vom,bo)) <= prvo AND
    P(GsetPosition(si,sim,vi,alpha,A,ai)) <= prsi AND
    P(GsetVelocity(vi,vim,bi)) <= prvi AND
    l = (norm(sm)+ao+ai+(alpha*d)*
      (norm(vm)+bo+bi))/(norm(vm)-bo-bi) AND
    psi = ao+ai+(min(T,l)+alpha*d)*(bo+bi) AND
    cd(D+psi,T+alpha*d,som,vom,sim,vim)=FALSE AND
    adsb_distr?(eta)(A)
    IMPLIES
      P({chi:Omega | conflict_rv(D,T,so,vo,si,vi)(chi) = True})
        <= prso+prvo+prsi+prvi+expt(1-eta,d+1)
```

4.3 Special Case: Absolute Bounds

The special case when absolute bounds on the positions and speeds of the ownship and the intruder are known and when there are no messages lost is considered next. That is, rather than letting $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i$ denote random variables, it is assumed in this section that these are positions and velocities, respectively (elements of $\mathbb{R}^2$). It is further assumed that there are no dropped ADS-B messages. Thus, each of the equations $\|\mathbf{s}_o - \mathbf{s}^{\mathbf{m}}_{\mathbf{o}}\| < a_o$, $\|\mathbf{s}_i - \mathbf{s}^{\mathbf{m}}_{\mathbf{i}}\| < a_i$, $\|\mathbf{v}_o - \mathbf{v}^{\mathbf{m}}_{\mathbf{o}}\| < b_o$, and $\|\mathbf{v}_i - \mathbf{v}^{\mathbf{m}}_{\mathbf{i}}\| < b_i$ is satisfied. In this case, Theorem 1 gives a safety buffer ψ for the separation distance D that ensure no missed alerts, assuming that there are no information delays such as dropped ADS-B messages. Thus, in the following corollary, each of the probabilities p_{so}, p_{vo}, p_{si}, and p_{vi} and the integer d, all occuring in the statement of Theorem 1, are zero.

Corollary 1 *Let* $\mathbf{s}^m = \mathbf{s}^{\mathbf{m}}_{\mathbf{o}} - \mathbf{s}^{\mathbf{m}}_{\mathbf{i}}$, $\mathbf{v}^m = \mathbf{v}^{\mathbf{m}}_{\mathbf{o}} - \mathbf{v}^{\mathbf{m}}_{\mathbf{i}}$. *Suppose that*

1. $\|\mathbf{v}^m\| > b_o + b_i$,
2. $\ell = (\|\mathbf{s}^m\| + a_o + a_i)/(\|\mathbf{v}^m\| - b_o - b_i)$,
3. $\psi = a_o + a_i + \min(T, \ell)(b_o + b_i)$, *and*
4. *conflict?*$(D, T, \mathbf{s}, \mathbf{v})$ *holds.*

Then, $\mathtt{cd}_{D+\psi,T}(\mathbf{s}^{\mathbf{m}}_{\mathbf{o}}, \mathbf{v}^{\mathbf{m}}_{\mathbf{o}}, \mathbf{s}^{\mathbf{m}}_{\mathbf{i}}, \mathbf{v}^{\mathbf{m}}_{\mathbf{i}}) = \mathtt{True}$.

Corollary 1 is proved in PVS by using Theorem 1 with α and d both equal to 0. The statement of that theorem depends on a probability space Ω, but it is true for any choice of Ω. To prove Corollary 1, the trivial probability space $(\Omega, \sigma(\Omega))$, where $\Omega = \{1\}$, $\sigma(\Omega) = \{\phi, \{1\}\}$, $P(\phi) = 0$, and $P(\{1\}) = 1$, is used.

It may be the case that instead of bounds on the measurement errors of the velocity vectors $\mathbf{v}_o$ and $\mathbf{v}_i$, bounds are known on the errors in the measurements of the ground speeds $\|\mathbf{v}_o\|$ and $\|\mathbf{v}_i\|$ and track angles $\mathit{track}(\mathbf{v}_o)$ and $\mathit{track}(\mathbf{v}_i)$ of the two aircraft. This may be the case when velocity information is broadcast not as a vector but as a track angle and ground speed pair. In this case, error bounds on track angles and ground speeds can be used to deduce error bounds on the velocity vectors themselves, thereby reducing this problem to that solved by Corollary 1.

Recall that the track angle $\mathit{track}(\mathbf{u})$ of a vector $\mathbf{u}$ is the angle $\alpha \in [0, 2\pi)$ that satisfies

$$\mathbf{u} = (\|\mathbf{u}\| \cos \alpha, \|\mathbf{u}\| \sin \alpha).$$

Here, ε_{so}, ε_{si}, $\varepsilon_{\alpha o}$, ε_{go}, $\varepsilon_{\alpha i}$, and ε_{gi} will denote the errors on the positions, track-angles, and ground speeds of the ownship and the intruder, respectively, i.e.,

$$\|\mathbf{s}_o - \mathbf{s_o^m}\| \leq \varepsilon_{so}, \tag{4.8}$$
$$\|\mathbf{s}_i - \mathbf{s_i^m}\| \leq \varepsilon_{si}, \tag{4.9}$$
$$|track(\mathbf{v}_o) - track(\mathbf{v_o^m})| \leq \varepsilon_{\alpha o}, \tag{4.10}$$
$$|\|\mathbf{v}_o\| - \|\mathbf{v_o^m}\|| \leq \varepsilon_{go}, \tag{4.11}$$
$$|track(\mathbf{v}_i) - track(\mathbf{v_i^m})| \leq \varepsilon_{\alpha i}, \tag{4.12}$$
$$|\|\mathbf{v}_i\| - \|\mathbf{v_i^m}\|| \leq \varepsilon_{gi}, \tag{4.13}$$

where ε_{so} and ε_{si} are strictly positive constants that denote the position error bounds for the ownship and intruder aircraft, respectively; $\varepsilon_{\alpha o}$ and $\varepsilon_{\alpha i}$ are strictly positive constants that denote the track error bounds for the ownship and intruder aircraft, respectively; and ε_{go} and ε_{gi} are strictly positive constants that denote the ground speed error bounds for the ownship and intruder aircraft, respectively.

Since $\varepsilon_{\alpha o}$, $\varepsilon_{\alpha i}$, ε_{go} and ε_{gi} are measurement errors, they are small compared to the measured values. Therefore, the following inequalities are assumed.

$$\begin{aligned} \varepsilon_{\alpha o} &\leq \frac{\pi}{2}, \\ \varepsilon_{go} &\leq \|\mathbf{v_o^m}\|, \\ \|\mathbf{v_o^m}\|(1-\cos\varepsilon_{\alpha o}) &\leq \varepsilon_{go}. \end{aligned} \tag{4.14}$$

$$\begin{aligned} \varepsilon_{\alpha i} &\leq \frac{\pi}{2}, \\ \varepsilon_{gi} &\leq \|\mathbf{v_i^m}\|, \\ \|\mathbf{v_i^m}\|(1-\cos\varepsilon_{\alpha i}) &\leq \varepsilon_{gi}. \end{aligned} \tag{4.15}$$

Velocity errors are given in terms of track error bounds, $\varepsilon_{\alpha o}$ for the ownship and $\varepsilon_{\alpha i}$ for the intruder, and ground speed error bounds, ε_{go} for the ownship and ε_{gi} for the intruder. However, as illustrated by Figure 1, velocity errors are also bounded by a circle.

The following lemma can be used to apply Corollary 1 in the case where error bounds on track angles and ground speeds are known instead of error bounds on the velocity vectors themselves.

Lemma 3 *Let $\mathbf{v}_o$, $\mathbf{v}_i$, $\mathbf{v_o^m}$, $\mathbf{v_i^m}$, $\varepsilon_{\alpha o}$, ε_{go}, $\varepsilon_{\alpha i}$, and ε_{gi} be such that they satisfy formulas* (4.10)–(4.15). *It holds that*

$$\begin{aligned} \|\mathbf{v}_o - \mathbf{v_o^m}\|^2 &\leq \varepsilon_{vo}^2, \\ \|\mathbf{v}_i - \mathbf{v_i^m}\|^2 &\leq \varepsilon_{vi}^2, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_{vo} &= \sqrt{2\|\mathbf{v_o^m}\|(\|\mathbf{v_o^m}\| + \varepsilon_{go})(1-\cos\varepsilon_{\alpha o}) + \varepsilon_{go}{}^2}, \\ \varepsilon_{vi} &= \sqrt{2\|\mathbf{v_i^m}\|(\|\mathbf{v_i^m}\| + \varepsilon_{gi})(1-\cos\varepsilon_{\alpha i}) + \varepsilon_{gi}{}^2}. \end{aligned}$$

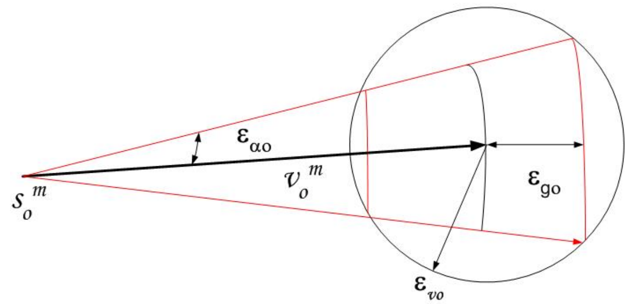

Figure 1: Ownship Velocity Error Bounds

5 Numerical Examples

DO-242A [23] specifies several system performance confidence-levels that are to be included in ADS-B messages detailing how precise and trusted the contained state information is. The relevant ones to this paper are the navigation accuracy categories for position and velocity (NAC_P and NAC_V). The 12 NAC_P categories define a maximum distance for errors in position (NAC_P 11 is $<$ 3 m, NAC_P 0 is $\geq$ 10 nmi); similarly the 5 NAC_V categories define maximum velocity error (NAC_V 4 is $<$ 0.3 m/s, NAC_V 0 is $\geq$ 10 m/s). That is, these numbers specify the parameters a_0, a_i and b_o, b_i, respectively. Both NAC_P and NAC_V specify that the stated values will fall within a 95% confidence interval, which is equivalent to saying that p_{so}, p_{vo}, p_{si} and p_{vi} are all equal to 0.05.

The ADS-B model described in Section 3.2 predicts that when aircraft are 60 nmi, $\eta \geq 0.95325$ (to 5 decimal places), while if they are 20 nmi apart, $\eta \geq 0.99996$.

Table 1 assumes both aircraft can produce data within the NAC_P 9 category (position error $<$ 30 m) and the NAC_V 4 category (velocity error $<$ 0.3 m/s). These numbers along with Equations (4.6) and (4.7) are used to compute the amount the distance that D needs to be increased, i.e., ψ, as well the associated upper bounds on the probabilities of missed alerts for varying choices of d. More accurate position data can reduce ψ by approximately 0.03 nmi, while less accurate data, especially velocity, can significantly increase the buffer ψ. Recall that, as above, d denotes the number of consecutive ADS-B messages from the intruder that were not received by the ownship, since the last received message from the intruder. The following table assumes that ADS-B updates from the aircraft are broadcast once per second ($\alpha = 1$ second). The relative ground speed $\|\mathbf{v}^m\| = 514$ m/s corresponds to two aircraft heading directly at each other, each traveling at approximately 500 knots. Furthermore, $\|\mathbf{v}^m\| = 206$ m/s corresponds to aircraft approaching each other at speeds of 200 knots.

Note that Equation (4.7) compensates for situations where the projected conflict is known to be less than the lookahead time (the l term). If both aircraft are 20 nmi from each other

T (s)	$\|\mathbf{s}^m\|$ (nmi)	$\|\mathbf{v}^m\|$ (m/s)	ψ (D buffer) ($\lambda = 0$ to 3 s)	$p_{missed-alert}$ $\lambda = 0$ s	$\lambda = 1$ s	$\lambda = 2$ s	$\lambda = 3$ s
300	60	514	0.10 nmi (190-193 m)	0.24675	0.20219	0.20010	0.20000
300	60	206	0.13 nmi (240-242 m)				
180	60	514	0.09 nmi (168-170 m)				
180	60	206	0.09 nmi (168-170 m)				
300	20	514	0.06 nmi (103-107 m)	0.20004	0.20000	0.20000	0.20000
300	20	206	0.09 nmi (168-172 m)				
180	20	514	0.06 nmi (103-107 m)				
180	20	206	0.09 nmi (168-170 m)				

Table 1: Lookahead, distance, relative speed, buffer sizes, and probability of missed alert

and are traveling at 500 knots, they could collide in as few as 72 seconds.

It should also be noted that the upper bounds on the probabilities of missed alerts in this table are high. However, this is not due to imprecision in the presented methods but to the fact that the confidence intervals specified in DO-242A are for 95% confidence and provide little knowledge of what is happening the other 5% of the time. These formulas could be used to calculate the probability of missed alerts that are significantly smaller if more precise confidence intervals were available for the positions and velocities of the aircraft.

6 Related Work and Conclusion

CD&R has been an area of active research since the last decade. In 2000, Kuchar and Yang [14] presented a taxonomy of conflict detection and resolution modeling methods that surveyed 68 different algorithms. One category in that taxonomy concerns the state propagation method. Probabilistic CD&R approaches use stochastic methods on predicted trajectory errors for estimating the probability of conflict or collision [2,18–20]. These methods are generally used in ground systems as they are often computational intensive.

Non-probabilistic CD&R methods such as those based on flight plans or linear state propagation use deterministic trajectory models. In these cases, safety buffers that increase the separation minima are used to reduce the number of false alarms. Gazit and Powell propose in [10] a separation standard based on the probability distribution functions of GPS and radar errors. In [25], Zhao presents a semi-analytical approach to determine appropriate separation minima between aircraft that takes into consideration wake-vortices and flight technical errors. The paper defines the uncertainty region as the difference between the measure and actual trajectories in an interval of time. The uncertainty region is an ellipsoid and the interval time is the maximum between the surveillance interval and the time needed for conflict avoidance. In [4], Consiglio et al. measured the impact of wind prediction to determine the additional safety buffer needed to preserve separation. The study is based on high-fidelity simulation. In the context of strategic conflict detection, Karr [12] describes different types of prediction error and proposes an algorithm to detect conflicts between trajectories that uses a notion of dynamic safety buffers.

This paper concerns safety buffers in state-based CD&R methods. These methods use the current state of the aircraft and a mass-point trajectory model (*nominal trajectories*, according to Kurchar and Yang's taxonomy) to alert a predicted violation of separation minima. In airborne concepts, state-based CD&R systems are used as backup of more advance separation assurance systems. For these kinds of systems, an approach for modeling aircraft state information uncertainty is proposed. The approach is illustrated with models of errors in GPS and ADS-B devices. However, other type of devices can be modeled in similar ways. These probabilistic models used to estimate the probability of a missed alert. From that estimation, analytical definitions of safety buffers are provided. These safety buffers guarantee that state-based conflict detection algorithms do not miss any alerts. Numerical examples of safety buffers for GPS and ADS-B parameters are given.

The analysis presented in this paper considers uncertainty in the current state information and a nominal trajectory that is a linear projection of this state. Therefore, trajectory uncertainties, such as navigation errors, are not part of the proposed uncertainty modeling approach. This simplification yields analytical definitions of safety buffers that are appropriate for state-based conflict detection. However, these safety buffers may be too conservative. Future work will consider a trajectory prediction model that uses previous state information of the aircraft.

The results presented in this paper have been mechanically checked using an interactive theorem prover (PVS), which provides strong guarantees that the mathematical development is correct. The use of a mechanical theorem prover requires a detailed description of the problem and a meticulous proof process. This level of rigor is justified by the critical role that aircraft separation plays in the overall safety of the next generation of air traffic management systems.

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